A Spherical Pythagorean Theorem

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There are probably many inequivalent statements in spherical geometry, somehow reducing to the Pythagorean theorem in the limit of an infinite radius of curvature $\mathtt{r}$. Among these, the Law of Cosines,

$$\cos(c/\mathtt{r}) = \cos(a/\mathtt{r})\cos(b/\mathtt{r}),$$

for a spherical right triangle with hypotenuse c and legs a and b, is generally presented as *the* 'spherical Pythagorean theorem'. Still, it has to be remarked that this formula does not have an immediate meaning in terms of areas of simple geometrical figures, as the Pythagorean theorem does. There is no diagram that can be drawn on the surface of the sphere to illustrate the statement in the spirit of ancient Greek geometry. In this note I reconsider the issue of extending the *geometrical* Pythagorean theorem to non-Euclidean geometries (with emphasis on the more intuitive spherical geometry).[1] In apparent contradiction with the statement that the Pythagorean proposition is equivalent to Euclid's parallel postulate, I show that such an extension not only exists, but also yields a deeper insight into the classical theorem.

The subject matter being familiar, I can dispense with preliminaries and start right in with Euclid's *Elements* [1].

The Pythagorean Theorem

The most celebrated theorem in mathematics [3] appears as Proposition 47 of Book I of Euclid's *Elements*. It says:

> In right-angled triangles the square on the side opposite to the right angle equals [the sum of] the squares on the sides containing the right angle.

The words 'the square *on* the side' refer to the area of the square constructed on the side, which only incidentally corresponds to 'the square *of* the side' in the sense of the second power of the length of the side. This correspondence no longer holds in spherical or hyperbolic geometry, generating not a little confusion about what the generalization of the theorem should be. On the other hand, since in Euclidean geometry the area of every regular polygon is proportional to the second power of the side, the change of preposition makes clear that the original Pythagorean squares can as well be replaced by equilateral triangles, regular pentagons, regular hexagons or any other kind of regular polygon. Equivalently, since the area enclosed by the circle is again proportional to the second power of the diameter/radius, the Pythagorean squares can also be replaced by circles with diameter/radius equal to the sides of the right triangle. The reach of the Pythagorean theorem can be extended even further. In Proposition 31 of Book VI of the *Elements*, Euclid himself states that we are actually free to replace the squares with arbitrary shapes provided they are similar:

> In right-angled triangles the figure on the side opposite to the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

[1] There is already a geometrical non-Euclidean generalization of the Pythagorean theorem [5], but it is not entirely satisfactory, because the figure on the hypotenuse is made to depend on the figures on the sides.

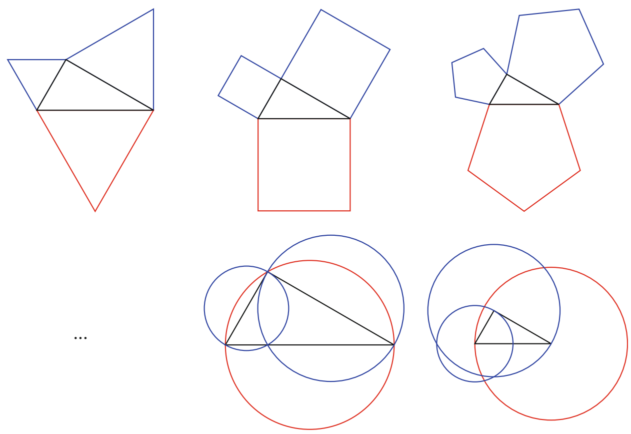

Figure 1. Diagrams representing some of the infinitely many equivalent variants of the Euclidean Pythagorean proposition.

We obtain infinitely many equivalent geometrical statements (see Figure 1), all summarized by the Pythagorean formula $c^2 = a^2 + b^2$, for any right triangle with hypotenuse c and legs a and b.

In spherical and hyperbolic geometry there is no concept of similar figures. The areas of regular polygons with equal sides are no longer proportional. Neither is the area of the circle proportional to that of a regular polygon with side equal to its diameter/radius or to that of another circle with radius equal to its diameter. All Pythagorean statements become inequivalent and none of them remains associated with the Pythagorean formula. The question we pose is whether at least one of these geometrical statements remains true when generalized to non-Euclidean geometries. Clearly, any generalization based on similarity is meaningless, but what about the ones linked by symmetry? To answer this question it is first necessary to decide what the generalization of right triangles, regular polygons, and circles is.

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For regular polygons and circles, the choice is somehow forced by symmetry. Not so for right triangles. The standard and apparently natural choice of identifying the class of *plane right triangles* with that of *spherical right triangles* is unsatisfactory in many respects. In Euclidean geometry the role of the right angle is unambiguous, and so is the distinction between hypotenuse and legs. In spherical geometry a triangle can have two or even three right angles—and, correspondingly, two 'hypotenuses' and three 'legs' or three 'hypotenuses' and three 'legs'. The very statement of the Pythagorean theorem makes little sense. If one persists in treating right triangles, the existence in spherical geometry of equilateral right triangles immediately provides a counterexample to all Pythagorean statements: The three figures constructed on the congruent sides are identical and the area of one of them can not equal the sum of the areas of the other two.

On the other hand, a plane right triangle can be characterized in many different ways. Just to mention the most obvious ones:

(*a*) a triangle with a right angle (whence the name);
(*b*) a triangle with an angle equal to a half of the sum of its interior angles;
(*c*) a triangle obtained by bisecting a rectangle (an *equiangular quadrilateral*, in preparation for non-Euclidean geometries) by means of its diagonal;
(*d*) an inscribed triangle having a diameter as a side.

Each characterization potentially provides a different generalization. The point is whether a generalization exists satisfying at least one of the infinitely many Pythagorean statements. To gain insight into this, let us briefly reconsider a few basic aspects of spherical geometry.

Spherical Triangles

Spherical geometry can be obtained by replacing Euclid's fifth postulate with the statement that no parallel to a given straight line can be drawn through a point not lying on it (in order to achieve a consistent system, however, the first and second postulates must also be partially modified). A model for such a geometry is the curved surface of a sphere of arbitrary radius $\mathtt{r}$. Straight lines are identified with great circles. On the sphere we can draw points, segments, angles, triangles, every kind of polygon and circles. Spherical triangles, in particular, come early on stage. They appear as Definition I of Book I of Menelaus's[2] *Sphaerica* [4]:

> A spherical triangle is the space included by arcs of great circles on the surface of a sphere.

The absence of a strong notion of parallelism on the sphere invalidates a number of important results of Euclidean geometry. Most remarkably, Proposition 32 of Book I of Euclid's *Elements* is replaced by:

[2]Menelaus of Alexandria (c. 70–140 CE) was the first to use arcs of great circles instead of parallel circles on the sphere. This marked a turning point in the development of spherical geometry. Being mainly interested in astronomical measurements and calculations, Menelaus did not consider theorems about area, like the Pythagorean theorem.

In any spherical triangle the sum of the three interior angles is greater than two right angles.

Thus, in spherical geometry (a) above is not equivalent to (b). This provides us with a first alternative generalization of plane right triangles to spherical geometry.

The difference between the sum of the interior angles and the straight angle

$$\varepsilon = \text{sum of interior angles} - \pi$$

is called the *spherical excess* of the triangle and is proved to be proportional to the area $\mathcal{A}$ of the triangle itself,

$$\mathcal{A} = \mathtt{r}^2\varepsilon.$$

By triangulation these results straightforwardly extend to every polygon: In any n-sided spherical polygon, the sum of the n interior angles is greater than $(2n-4)$ right angles, and the area of the polygon equals $\mathtt{r}^2$ times its spherical excess. In particular, the sum of the four congruent interior angles of a spherical square is greater than four right angles. Hence, these angles are no longer right. The triangulation of a spherical square by means of its diagonal no longer produces two right triangles. The same holds for every equiangular quadrilateral. It follows that (a) is not equivalent to (c). This provides us with a second possible generalization of plane right triangles to spherical geometry.

A third possibility comes from the failure of Proposition 20 of Book III of Euclid's *Elements* and of its corollaries. In particular:

In a given spherical circle, all inscribed angles subtending the diameter are greater than a right angle.

Inscribed angles subtending the diameter are no longer right. Therefore, in spherical geometry (a) is not equivalent to (d).

Quite remarkably, in spherical geometry (b), (c), and (d) are equivalent.

To see that (c) implies (b), consider Figure 2. Since equiangular quadrilaterals have opposite sides congruent, ABC and ACD are congruent. Denote by ε their spherical excess. Since spherical excess is proportional to the area and the area of the equilateral quadrilateral $ABCD$ equals the sum of the areas of the triangles ABC and ACD, the spherical excess of the equiangular quadrilateral equals 2ε. The sum of its interior angles is therefore $2\pi + 2\varepsilon$. Given the congruence of the four interior angles, we obtain

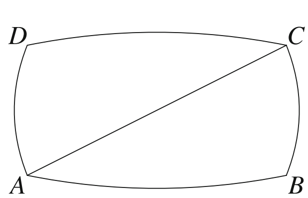

Figure 2. Spherical triangles obtained by dividing an equiangular quadrilateral by means of its diagonal are not right-angled.

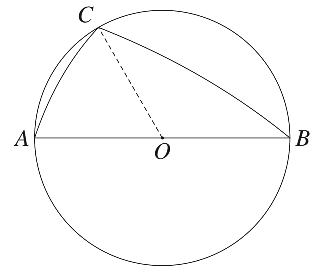

Figure 3. Inscribed spherical triangles having a diameter as a side are not right-angled.

$$\angle ABC = \frac{\pi+\varepsilon}{2}.$$

To prove the opposite implication, we just double a spherical triangle ABC with $\angle ABC = \frac{\pi+\varepsilon}{2}$ and join the two copies along AC with A and C interchanged. Since $\angle BAC + \angle ACB = \frac{\pi+\varepsilon}{2}$ we obtain an equiangular quadrilateral.

To see that (d) implies (b), denote by ε the spherical excess of the triangle ABC in Figure 3. Draw the segment OC dividing ABC into two isosceles triangles AOC and BOC. Denote by ε_1 the spherical excess of the first one and by ε_2 that of the second one. Clearly, $\varepsilon = \varepsilon_1 + \varepsilon_2$. Since $\angle OCA \equiv \angle OAC$, from the first triangle, we obtain

$$2\angle OCA + \angle AOC = \pi + \varepsilon_1,$$

and since $\angle OCB \equiv \angle OBC$, from the second one we have

$$2\angle OCB + \angle BOC = \pi + \varepsilon_2.$$

Adding term by term, recalling that $\angle ACO + \angle BCO \equiv \angle ACB$ and $\angle AOC + \angle BOC = \pi$, we obtain

$$\angle ACB = \frac{\pi+\varepsilon}{2}.$$

Finally, to prove that (b) implies (d), we consider a spherical triangle ABC with $\angle ACB = \frac{\pi+\varepsilon}{2}$. We now choose point O on AB such that $\angle ACO$ equal to $\angle BAC$. Thus, $CO \equiv AO$. At this point, we observe that $\angle BCO = \frac{\pi+\varepsilon}{2} - \angle ACO = \frac{\pi+\varepsilon}{2} - \angle BAC = \angle CBA$. Thus, $CO \equiv BO$, and the triangle ABC is inscribed in a circle with diameter AB.

The transition from Euclidean to spherical geometry seems to preserve the property of 'having one angle equal to a half of the sum of its interior angles' and not the property of 'having a right angle'. This provides us with a promising class of triangles generalizing plane right triangles to non-Euclidean geometries. Let us therefore introduce a suitable terminology:

We say that a triangle is *properly angled*, or, equivalently, that it is a *proper triangle*, when it has an angle equal to a half of the sum of its interior angles. That angle is called the *proper angle* of the triangle; the side opposite to it, the *hypotenuse*; and the sides containing it the *legs*.

The role of the proper angle is unambiguous, and so is the distinction between hypotenuse and legs. In plane geometry the class of proper triangles corresponds to that of right triangles. In spherical geometry the class of proper triangles shares at least some of the fundamental properties enjoyed by plane right triangles: Any equiangular quadrilateral is divided by means of its diagonal into two proper triangles; an inscribed triangle having as side a diameter is a proper triangle. It is then natural to wonder whether spherical proper triangles enjoy at least one of the infinitely many symmetric variants of the Pythagorean proposition. Recalling the formula expressing the area of a spherical regular polygon of side l,

$$\mathcal{A}_{n-\text{gon}} = 2\pi \mathtt{r}^2 - 2n\mathtt{r}^2 \sin^{-1}\sqrt{\frac{\cos(l/\mathtt{r}) - \cos(2\pi/n)}{\cos(l/\mathtt{r}) + 1}}$$

and the formula for the area of a spherical circle of radius r

$$\mathcal{A}_{\text{circle}} = 2\pi\mathtt{r}^2(1 - \cos(r/\mathtt{r})),$$

we can simply proceed to a direct check of all of them. It is a wonderful surprise to discover that one of them still holds true.

Pythagoras on the Sphere ...

To pay homage to ancient Greek geometers, we state the proposition as follows:

> In properly angled triangles, the circle on the side opposite to the proper angle equals [the sum of] the circles on the sides containing the proper angle.

Here, the words 'the circle on the side' mean the area of the circle having the side as radius; this time there is no risk of algebraic confusion.

The proposition is illustrated by the beautiful diagram of Figure 4. It is also immediate how to give an analytical proof of it. Parametrizing the sphere by standard spherical coordinates θ and ϕ, we consider an arbitrary equiangular quadrilateral $ABCD$ centered at the north pole and with diagonal on the great circle through the pole and (1, 0). Its vertices lie at $A(\hat\theta, 0)$, $B(\hat\theta, \hat\phi)$, $C(\hat\theta, \pi)$, $D(\hat\theta, \hat\phi - \pi)$, for some angles $\hat\theta$ and $\hat\phi$. Given the equivalence of (b) and (c), ABC is an arbitrary proper triangle. By means of the spherical distance formula for generic points $P(\theta_P, \phi_P)$ and $Q(\theta_Q, \phi_Q)$,

$$\overline{PQ} = \mathtt{r}\cos^{-1}\left[\cos\theta_P\cos\theta_Q + \sin\theta_P\sin\theta_Q\cos(\phi_Q - \phi_P)\right],$$

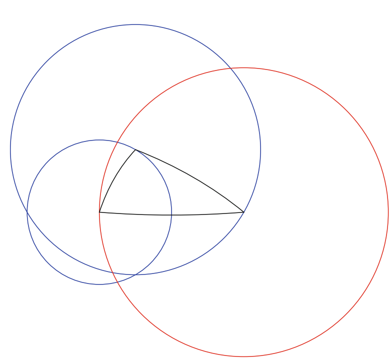

Figure 4. The spherical Pythagorean proposition.

we evaluate the lengths of the sides as

$$\begin{aligned}\overline{AB} &= \mathtt{r}\cos^{-1}(\cos^2\hat\theta + \sin^2\hat\theta\cos\hat\phi),\\ \overline{BC} &= \mathtt{r}\cos^{-1}(\cos^2\hat\theta - \sin^2\hat\theta\cos\hat\phi),\\ \overline{AC} &= \mathtt{r}\cos^{-1}(\cos^2\hat\theta - \sin^2\hat\theta).\end{aligned}$$

Dividing by $\mathtt{r}$ and taking the cosine of the resulting expressions we have

$$\begin{aligned}\cos(\overline{AB}/\mathtt{r}) &= \cos^2\hat\theta + \sin^2\hat\theta\cos\hat\phi,\\ \cos(\overline{BC}/\mathtt{r}) &= \cos^2\hat\theta - \sin^2\hat\theta\cos\hat\phi,\\ \cos(\overline{AC}/\mathtt{r}) &= \cos^2\hat\theta - \sin^2\hat\theta.\end{aligned}$$

Adding the first two equalities and comparing the result with the third one, after a very little algebra we obtain

$$\begin{aligned}2\pi\mathtt{r}^2(1 - \cos(\overline{AC}/\mathtt{r})) &= 2\pi\mathtt{r}^2(1 - \cos(\overline{AB}/\mathtt{r}))\\ &\quad + 2\pi\mathtt{r}^2(1 - \cos(\overline{BC}/\mathtt{r})).\end{aligned}$$

Recalling the formula for the area of the spherical circle in terms of its radius, we recognize the spherical Pythagorean proposition. Clearly, in the limit of a large radius of curvature $\mathtt{r}$, this expression reduces to the Pythagorean formula $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.

... and on the Hyperbolic Plane

The proposition straightforwardly extends to the less intuitive hyperbolic geometry. This is proved pretty much in the same way. As hyperbolic plane model we consider the quadric

$$x^2 + y^2 - z^2 = -\mathtt{r}^2$$

embedded in the Minkowskian space $\mathbf{R}^{2,1}$. By introducing hyperbolic polar coordinates

$$\vec{x} = (\mathtt{r}\sinh w\cos\phi, \mathtt{r}\sinh w\sin\phi, \mathtt{r}\cosh w),$$

the plane is parametrized by the hyperbolic latitude w, $w \geq 0$, and by the longitude ϕ, $-\pi < \phi \leq \pi$. The distance formula for generic points $P(w_P, \phi_P)$, $Q(w_Q, \phi_Q)$ reads

$$\begin{aligned}\overline{PQ} &= \mathtt{r}\cosh^{-1}\\ &\quad\left[\cosh w_P\cosh w_Q - \sinh w_P\sinh w_Q\cos(\phi_Q - \phi_P)\right].\end{aligned}$$

As in spherical geometry, proper triangles are obtained by dividing equiangular quadrilaterals by means of their diagonals. Hence, we again consider an arbitrary equiangular quadrilateral $ABCD$ centered at the pole (0, 0), with diagonal along the hyperbolic line through the pole and (1, 0). The vertices lie at $A(\hat w, 0)$, $B(\hat w, \hat\phi)$, $C(\hat w, \pi)$, $D(\hat w, \hat\phi - \pi)$, for some values $\hat w$ and $\hat\phi$. ABC is an arbitrary proper triangle. The lengths of its sides are evaluated as

$$\begin{aligned}\overline{AB} &= \mathtt{r}\cosh^{-1}(\cosh^2\hat w - \sinh^2\hat w\cos\hat\phi),\\ \overline{BC} &= \mathtt{r}\cosh^{-1}(\cosh^2\hat w + \sinh^2\hat w\cos\hat\phi),\\ \overline{AC} &= \mathtt{r}\cosh^{-1}(\cosh^2\hat w + \sinh^2\hat w).\end{aligned}$$

Dividing by $\mathtt{r}$, taking the hyperbolic cosine of the three expressions, and recalling the identity $\sinh^2 x = \cosh^2 x - 1$, after some algebra we obtain

$$2\pi\mathtt{r}^2(\cosh(\overline{AC}/\mathtt{r}) - 1) = 2\pi\mathtt{r}^2(\cosh(\overline{AB}/\mathtt{r}) - 1) + 2\pi\mathtt{r}^2(\cosh(\overline{BC}/\mathtt{r}) - 1).$$

Recalling the formula for the area of an hyperbolic circle of radius r

$$\mathcal{A}_{\text{circle}} = 2\pi\mathtt{r}^2(\cosh(r/\mathtt{r}) - 1),$$

we recognize the hyperbolic Pythagorean proposition. The Euclidean Pythagorean formula is again obtained in the limit of a large radius of curvature $\mathtt{r}$.

Epilogue

The Pythagorean theorem is generally claimed to be equivalent to Euclid's fifth postulate. If so, then it can hold only in Euclidean geometry. As we have seen in this paper, this very much depends on how the proposition is understood. If we insist on squares on the sides of right triangles, no doubt the claim is true. Nevertheless, if we take a slightly wider viewpoint by considering all the equivalent variants of the theorem, and classes of triangles that better embody the properties of plane right triangles in non-Euclidean geometry, we come to a statement that equally holds in Euclidean, spherical, and hyperbolic geometry.

Since it is true in Euclidean and hyperbolic geometry, this statement belongs to neutral geometry. In principle, it could be included among the first 28 propositions of the *Elements* and should be capable of a proof in terms of the first four Euclidean postulates. Since it is also true in spherical geometry, the statement should actually follow from an even smaller set of axioms. In any case, it represents a more basic theorem about area than the original Pythagorean theorem (as in Euclidean geometry, spherical and hyperbolic polygons of the same area are related by scissor congruence [2]).

In this paper we presented an analytical proof of the spherical and hyperbolic Pythagorean propositions. In the final analysis, this proof follows from the Euclidean Pythagorean proposition itself. It goes without saying that a synthetic proof based on a minimal choice of postulates would be of great interest.

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